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# MAXIMUM THEOREMS FOR CONVEX STRUCTURES WITH AN APPLICATION TO THE THEORY OF OPTIMAL INTERTEMPORAL ALLOCATION*

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A mathematical result often used in economic theory, is Berge's Maximum Theorem. This establishes continuity of the value function and upper semicontinuity of the maximizers' correspondence. However, the theorem requires the return function and the feasible correspondence to be continuous. For some applications in economics, it is difficult to justify these strong continuity requirements but quite possible to explain some 'convex structures' to the problem. The main purpose of this paper is to present a maximum theorem under convex structures but with weaker continuity requirements. We then illustrate the usefulness of our results by an application to a problem encountered in the theory of optimal intertemporal allocation.

## 1. Introduction

The following situation is often encountered in many problems of economic theory. There is a set $X$ of possible states. For each state $x$ in $X$, there is a set $g(x)$ [a subset of $Y$] of actions available to an agent. The agent gets a return $f(y)$ if he can pick action $y$ in $Y$. The agent is then interested, given $x$ in $X$, in maximizing $f(y)$ subject to the restriction that $y$ be in $g(x)$. Under this set-up, one is often interested in knowing whether the *maximized value* of the function, $f$, is continuous on $X$; also, whether the correspondence of *maximizing actions* has some continuity properties. The maximum theorem of Berge (1963) provides suitable answers to this set of questions. There are several generalizations and useful expositions of this result (with economic applications in mind) in Debreu (1959, 1969), Sonnenschein (1971), Hildenbrand (1974) and Walker (1979). [For a survey of results (from a different perspective), and for references to the rather large literature on the subject, see Bank et al. (1983)].

The maximum theorem of Berge requires that the return function be continuous, and that the feasible actions' correspondence, $g$, be also conti-

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nuous. For some applications in economics, it is difficult to justify these strong continuity requirements, but quite possible to justify some 'convex structure' to the problem [such as $X$ and $Y$ are convex, $f$ is concave, and $g$ has a convex graph]. The main purpose of this paper is to present a maximum theorem under weaker continuity requirements on $f$ and $g$, but with the type of convex structure we mentioned above. We then illustrate the usefulness of our result by applying it to a problem encountered in the theory of optimal intertemporal allocation.

## 2. Preliminaries

### 2.1. Notation

Let $R^m$ be an $m$-dimensional real space. The norm on $R^m$ is the Euclidean one denoted by $\|\cdot\|$. For any $x, y$ in $R^m$, $x \geqq y (x \gg y)$ means $x_i \geqq y_i (x_i > y_i)$ for $i = 1, \ldots, m$; $x > y$ means $x \geqq y$ and $x \neq y$. The set $\{x \text{ in } R^m\colon x \geqq 0\}$ is denoted by $R^m_+$.

Let $N$ be the set of non-negative integers $\{0, 1, 2, \ldots\}$. Let $S = \{\boldsymbol{x} = \{x_s\}_0^\infty\colon x_s$ is in $R^m$ for all $s$ in $N\}$. Thus, $S$ is the space of sequences of vectors in $R^m$. The metric, $d$, on $S$ is defined as follows. For $\boldsymbol{x}$, $\boldsymbol{y}$ in $S$,

$$d(\boldsymbol{x}, \boldsymbol{y}) = \sum_{s=0}^{\infty} (1/2^s)\{\|x_s - y_s\|/[1 + \|x_s - y_s\|]\}.$$

It is known that $S$ with the above metric defines a metric linear space and that $d(\boldsymbol{x}^\tau, \boldsymbol{x}) \to 0$ iff $\|x_s^\tau - x_s\| \to 0$ for all $s$ in $N$.

For any set $Y$, let $P(Y)$ denote the collection of all subsets of $Y$. A correspondence, $g$, from a set $X$ to a set $Y$ is then a mapping from $X$ to $P(Y)$, and we write $g\colon X \to P(Y)$ to denote this fact.

### 2.2. Definitions

In the rest of the paper, $X$ will be a subset of $R^m$, and $Y$ a subset of a metric linear space, $Y$, with metric $|\cdot|$.

Let $\bar{x}$ belong to $X$. (i) A correspondence $g\colon X \to P(Y)$ is *lower hemicontinuous at* $\bar{x}$ iff '$\|x^n - \bar{x}\| \to 0$ as $n \to \infty$, and $\bar{y}$ *in* $g(\bar{x})$' imply that 'there is an integer $\bar{n}$, and a sequence $y^n$ in $g(x^n)$ for $n \geqq \bar{n}$, such that $|y^n - \bar{y}| \to 0$ as $n \to \infty$'. (ii) A correspondence $g\colon X \to P(Y)$ is *upper hemicontinuous at* $\bar{x}$ if $g(\bar{x})$ is non-empty and compact, and '$\|x^n - \bar{x}\| \to 0$, $y^n$ in $g(x^n)$ for all $n$' imply that 'there is a converging subsequence of $\{y^n\}_0^\infty$, whose limit belongs to $g(\bar{x})$'. (iii) The correspondence $g\colon X \to P(Y)$ is *continuous at* $\bar{x}$ if it is both upper and lower hemicontinuous at $\bar{x}$.

The correspondence $g\colon X \to P(Y)$ is lower (upper) hemicontinuous on $X$, if it is lower (upper) hemicontinuous at each point of $X$. It is continuous on $A$ if it is both upper and lower hemicontinuous on $X$.

The *graph* of the correspondence $g:X\to P(Y)$ is the set $\{(x,y)$ in $X\times Y: y$ is in $g(x)\}$. The correspondence $g:X\to P(Y)$ is *monotone increasing* if $x'\geqq x$ implies $g(x')\supset g(x)$.

A function, $f:Y\to R$ is said to be *upper semicontinuous* at $\bar{y}$ in $Y$ if '$|y^n-\bar{y}|\to 0$ as $n\to\infty$' implies '$\lim_{n\to\infty}\sup f(y^n)\leqq f(\bar{y})$'. It is *upper semicontinuous on* $Y$ if it is upper semicontinuous at each $y$ in $Y$. A function, $f:Y\to R$ is *lower semicontinuous* at $\bar{y}$ (on $Y$) if $[-f]$ is upper semicontinuous at $\bar{y}$ (on $Y$). It is *continuous* at $\bar{y}$ (on $Y$) if it is both upper and lower semicontinuous at $\bar{y}$ (on $Y$).

## 3. Maximum theorems

The following type of problem is often encountered in many branches of economic theory. There is a set $X$ of possible *states*. For each state $x$ in $X$, there is a set $g(x)\subset Y$ of *actions* available to an agent. The agent has a *return function*, $f$, which tells him the return $f(y)$ that he will get, if he can pick the action $y$ in $Y$. The agent is then interested in maximizing $f(y)$ subject to the restriction that $y$ be in $g(x)$.

Now, given some properties of $X$, $Y$, $g$ and $f$, one is often interested in knowing whether the *maximum value*, $M(x)$, of the function $f$, defined by $M(x)=\max\{f(y): y$ is in $g(x)\}$ is a continuous function on $X$. One is also interested in knowing whether the *correspondence of maximizers*, defined by $h(x)=\{y: y$ is in $g(x)$, and $f(y)=M(x)\}$ has some continuity properties. The answers to these questions are provided by the so-called 'Maximum theorem'.

We first state here, for ready reference, the Maximum Theorem of Berge (1963, p. 116).

*Theorem 1* (*Berge*). *If $f: Y\to R$ is a continuous function, and $g: X\to P(Y)$ is a continuous correspondence, then the maximum value, $M$, is continuous on $X$, and the correspondence of maximizers, $h$, is upper hemicontinuous on $X$.*

We are primarily interested in establishing maximum theorems in which weaker *continuity* requirements are imposed on the function $f$, and the correspondence, $g$. Instead, the framework has a 'convex structure' that can be justified in some applications in economics. To elaborate, the set of states, $X$, will be a *convex* subset of $R^m$; the set of actions will be a *convex* subset $Y$, of the metric linear space, $Y$, with metric $|\cdot|$. The feasible correspondence of actions, $g:X\to P(Y)$, will have a *convex* graph, and the return function, $f:Y\to R$ will be *concave*. On the other hand, $f$ is only assumed to be *upper semicontinuous* on $Y$, and $g$ is only assumed to be an *upper hemicontinuous* correspondence on $X$. However, it is not possible to establish a maximum theorem with the above structure as the folowing example shows.

*Example 1.* Let

$$X=\{(x_1,x_2) \text{ in } R^2_+ : \|(x_1,x_2)\| \leqq 1\};\ Y=\{(y_1,y_2) \text{ in } R^2_+ : \|(y_1,y_2)\| \leqq \sqrt{2}\}.$$

Let $G: X \to P(Y)$ be defined by $G(x_1,x_2)=\{(y_1,y_2)$ in $R^2_+ : (y_1,y_2) \leqq (1,1)$ for $(x_1,x_2)=(1,0)$ and $(y_1,y_2) \leqq (0,1)$ for $(x_1,x_2) \neq (1,0)\}$. Let $g: R^2_+ \to P(R^2_+)$ be defined by $g(x_1,x_2)=\{(y_1,y_2)$ in $R^2_+ : (x_1,x_2,y_1,y_2)$ is in the convex hull of the graph of $G\}$. Since $X$ is convex, $g$ is non-empty only on $X$, and so $g$ maps $X$ to $P(Y)$. Let $f: Y \to R$ be defined by $f(y_1,y_2)=y_1+y_2$.

It is easy to check that $g$ is upper hemicontinuous on $X$. Furthermore, the graph of $g$ is convex, and $f$ is continuous and concave on $Y$. However, $h$ is *not* upper hemicontinuous and $M$ is not continuous on $X$. To establish this, note that if $x$ is in $X$ with $\|x\|=1$ then $x=\sum_{r=1}^{n} \lambda_r x^r$ for $x^r$ in $X$, $\lambda_r \geqq 0$ for $r=1,\ldots,n$, and $\sum_{r=1}^{n} \lambda_r = 1$. This implies $x^r = x$ whenever $\lambda_r > 0$. [This is the assertion that points on the unit circle cannot be expressed as a convex combination of points in it]. Now, pick a sequence $(x_1^n, x_2^n) = ([n^2-1)/n]^{\frac{1}{2}}, 1/n)$ for $n=1,2,\ldots$. Then $(x_1^n,x_2^n)$ converges to $(1,0)$ as $n \to \infty$. For $(y_1^n,y_2^n)$ in $g(x_1^n,x_2^n)$, we must have $(y_1^n,y_2^n)$ in $G(x_1^n,x_2^n)$, and so $(y_1^n,y_2^n) \leqq (0,1)$ for all $n$. So $(y_1^n,y_2^n)$ in $h(x_1^n,y_2^n)$ implies $(y_1^n,y_2^n)=(0,1)$, given the definition of $f$. Similarly, $(y_1,y_2)$ in $h(1,0)$ implies $(y_1,y_2)=(1,1)$. Clearly then, $h$ is not upper hemicontinuous at $(1,0)$, and $M$ is not continuous at $(1,0)$.

Example 1 shows that we need some *additional* structure (besides convexity) to prove a maximum theorem under weaker continuity requirements. One such set of restrictions is conveyed in the condition stated below.

*Condition A.* $g$ is monotone increasing, and $0 \in X \subset R^m_+$.

First, we state and prove a maximum theorem under Condition A. Then, we remark briefly on some alternative restrictions under which similar maximum theorems can be proved.

*Theorem 2. Let $X$ be a non-empty, convex subset of $R^m$, and $Y$ be a non-empty, convex subset of $Y$. Let $g$ be an upper hemicontinuous correspondence from $X$ to $Y$, with a convex graph. Let $f: Y \to R$ be upper semicontinuous and concave. If Condition A is satisfied, then $M$ is continuous and concave, and $h$ is upper hemicontinuous and convex-valued on $X$.*

*Proof.* Since $g(x)$ is non-empty and compact for each $x$ in $X$, and $f$ is upper semicontinuous, so $M$ and $h$ are well defined on $X$. Further, since $f$ is concave and $g$ has a convex graph, $h$ is convex-valued, and $M$ is concave. By the upper semicontinuity of $f$ and the upper hemicontinuity of $g$, $M$ is upper semicontinuous on $X$ [Berge (1963, p. 116, Theorem 2)].

To establish continuity of $M$, we then have to show that $M$ is lower semicontinuous. Note first that $g$ monotone increasing implies that $M$ is monotone increasing; that is, for $x' \geqq x$, $M(x') \geqq M(x)$. The monotonicity and concavity of $M$, together with the hypothesis that $0 \in X \subset R^m_+$, implies that $M$ is lower semicontinuous. To see this, let $\{x^n\}_{n=0}^{\infty}$ be any sequence converging to $\bar{x}$ in $X$. For any $\varepsilon > 0$, let $\lambda$ in $(0, 1)$ be such that $\lambda M(\bar{x}) + (1-\lambda)M(0) \geqq M(\bar{x}) - \varepsilon$. Since $\{x^n\}_{n=0}^{\infty}$ is a sequence converging to $\bar{x}$ there is some integer $\bar{n}$, such that $x^n \geqq \lambda\bar{x}$, for all $n \geqq \bar{n}$. From the monotonicity and concavity of $M$, $M(x^n) \geqq M(\lambda\bar{x}) \geqq \lambda M(\bar{x}) + (1-\lambda)M(0)$. From the choice of $\lambda$, we can therefore conclude that $M(x^n) \geqq M(\bar{x}) - \varepsilon$, for all $n \geqq \bar{n}$. Hence $\liminf_{n\to\infty} M(x^n) \geqq M(\bar{x}) - \varepsilon$. Since $\varepsilon > 0$ was arbitrarily chosen, it must be the case that $\liminf_{n\to\infty} M(x^n) \geqq M(\bar{x})$.

To demonstrate the upper hemicontinuity of $h$, define $H(x) = \{y$ in $Y: f(y) - M(x) \geqq 0\}$. From the continuity of $M(x)$ and the upper semicontinuity of $f(y)$, it follows that $H$ has a closed graph. So, $h(x) = H(x) \cap g(x)$ is upper hemicontinuous [by Berge (1963, p. 112, Theorem 7)]. □

*Remark 1.* (i) In Theorem 2, instead of assuming that 0 is in $X$ and $X \subset R^m_+$, one can also assume that for every $x$ in $X, X \subset R^m$, there is $x'$ in $X$, such that $x' \ll x$. The conclusions remain unaltered and the proof is almost identical [see McKenzie (1986)].
(ii) If the return function, $f$, is defined on $X \times Y$, then a maximum theorem like Theorem 2 can be proved, if $f$ is upper semicontinuous and concave on $X \times Y$, and monotone non-decreasing in $x$, for all $y$ in $Y$.

*Remark 2.* (i) In Theorem 2, suppose we replace Condition A by the following condition:

*Condition B.* $X$ is a locally simplicial set.

Then the conclusions of the theorem continue to hold. This can be seen by noting that [following Rockafellar (1970, p. 84)] a concave function is lower semicontinuous on a locally simplicial set, and then applying this fact to the function, $M$, in our proof. [The definition of a 'locally simplicial set', as well as the above result, can be found in Dutta and Mitra (1985).] Furthermore, this result can be easily generalized to the case where $f$ is defined on $X \times Y$, provided $f$ is upper semicontinuous and concave on $X \times Y$.
(ii) The class of locally simplicial sets includes simplices, polytopes and polyhedral sets in $R^m$. Hence a number of standard parameter sets encountered in economic theory belong to this class.

*Remark 3.* Example 1 demonstrated that graph $g$ being convex does not imply that $g$ is lower hemicontinuous. However, under the convex structure of Theorem 2 and either Condition A or Condition B, $g$ can in fact be shown to be lower hemicontinuous. Theorem 1 continues to be inapplicable, since $f$ may not be continuous. Theorem 2 and Remark 2 demonstrate that the available sufficient conditions for continuity of $g$ can be used together with the concavity and upper semicontinuity of $f$ to ensure the continuity and concavity of $M$, and the convex-valuedness and upper hemicontinuity of $h$.

A relevant question that one may ask is whether the continuity of $M$ and upper hemicontinuity of $h$ can be proved if concavity of $f$ is weakened to quasi-concavity in Theorem 2. The following example shows that this extension is not possible.

*Example 2.* Let $X=[0,1]=Y$, $g(x)=[1-x,1]$ for $x$ in $X$; $f(y)=\frac{1}{2}$ for $y>0$, and $f(y)=1$ for $y=0$. Then, all the hypotheses of Theorem 2 are satisfied [and, in fact, both Conditions A and B hold], except that $f$ is quasi-concave, but not concave. Note that $M(x)=\frac{1}{2}$ for all $0\leqq x<1$; $M(1)=1$. Hence, for $x^n=1-(1/n)$, $n=1,2,\dots,h(x)$ contains 1 but $h(1)$ does not contain 1. So $h$ is not upper hemicontinuous, and $M$ is not continuous.

Another question of interest in whether the correspondence of maximizers, $h$, can be shown to be lower hemicontinuous. It is easy to construct an example which shows that this is not possible even when all the hypotheses of Theorems 1 *and* 2 (and even Condition B) are satisfied [see Dutta and Mitra (1985)].

In some instances, a weaker maximality notion is used, namely that of $\varepsilon$-maximality. An $\varepsilon$-maximizer correspondence can be defined as $h_\varepsilon(x)=\{y$ in $Y{:}y$ is in $g(x)$ and $f(y)>M(x)-\varepsilon\}$. Under the hypotheses of Theorem 1, one can show, following Majumdar (1983), that for any $\varepsilon>0$, $h_\varepsilon(x)$ is lower hemicontinuous. It can also be proved that for $\varepsilon>0$, $h_\varepsilon(x)$ is lower hemicontinuous, when the hypotheses of Theorem 2 [or those of Remark 2] are satisfied.

## 4. An application to optimal intertemporal allocation theory

In this section, we provide an application of Theorem 2 to a problem in optimal intertemporal allocation theory. We show that the 'value function' associated with the typical dynamic optimization problem is continuous, and the 'optimal policy correspondence' is upper semicontinuous.

A standard framework of optimal intertemporal allocation can be described in the following way. The *economy E* consists of a triple $(\Omega, u, \delta)$, where $\Omega\subset R^m_+\times R^m_+$ is the *technology* set, $u{:}\Omega\to R$ is the *utility function*, and

$0<\delta<1$ is a *discount factor*. [For details, see Khan and Mitra (1986).] The following assumptions are maintained on $E$:

*A.1.* (i) $(0,0)$ is in $\Omega$, and (ii) if $(0,y)$ is in $\Omega$, then $y=0$.

*A.2.* $\Omega$ is closed and convex.

*A.3.* If $(x,y)$ is in $\Omega$, and $x'\geqq x$, $0\leqq y'\leqq y$, then $(x',y')$ is in $\Omega$, and $u(x',y')\geqq u(x,y)$.

*A.4.* There is $\beta>0$, such that $\|x\|>\beta$, and $(x,y)$ in $\Omega$ implies that $\|y\|\leqq\|x\|$.

*A.5.* $u$ is upper semicontinuous and concave in $\Omega$.

*A.6.* There is $\alpha$ in $R$, such that $(x,y)$ in $\Omega$ implies $u(x,y)\geqq\alpha$.

A *program* from $k$ in $R^m_+$ is a sequence $\{k(t)\}_{t=0}^{\infty}$ such that $k(0)=\bar{k}$ and $(k(t),\ k(t+1))$ is in $\Omega$ for $t\geqq 0$. A program $\{\hat{k}(t)\}_{t=0}^{\infty}$ from $\bar{k}$ is an *optimal program* from $\bar{k}$ if $\sum_{t=0}^{\infty}\delta^t u(kt),\ k(t+1))\leqq\sum_{t=0}^{\infty}\delta^t u(\hat{k}(t),\ \hat{k}(t+1))$ for every program $\{k(t)\}_{t=0}^{\infty}$ from $k$. It is a *stationary optimal program* if it is an optimal program, and $\hat{k}(t)=\bar{k}$ for $t\geqq 0$. A *stationary optimal stock* $\bar{k}$ is an element of $R^m_+$, such that $\{\bar{k}\}_{t=0}^{\infty}$ is a stationary optimal program. It is non-trivial if $u(\bar{k},\bar{k})>u(0,0)$.

Define for each $k$ in $R^m_+$, $B(k)=\max\{\|k\|,\beta\}$. It is well known that if $\{k(t)\}_{t=0}^{\infty}$ is a program from $k$, then $\|k(t)\|\leqq B(k)$ for $t\geqq 0$, and furthermore, by A.5, A.6, $\sum_{t=0}^{\infty}\delta^t u(k(t),k(t+1))$ is absolutely convergent. It can be shown that for each $k$ in $R^m_+$, there exists an optimal program $\{\hat{k}(t)\}_{t=0}^{\infty}$ from $k$. Define the *value function*, $V$, from $R^m_+$ to $R$ by $V(k)=\max[\sum_{t=0}^{\infty}\delta^t u(k(t),k(t+1)):\{k(t)\}_{t=0}^{\infty}$ is a program from $k]$. Then, if $\{\hat{k}(t)\}_{t=0}^{\infty}$ is an optimal program from $k$, we have $V(k)=\sum_{t=0}^{\infty}\delta^t u(\hat{k}(t),\hat{k}(t+1))$. We can also define the correspondence $\psi(k)=\{\{k(t)\}_{t=0}^{\infty}$: $\{k(t)\}_{t=0}^{\infty}$ is an optimal program from $k\}$.

*Proposition 1. V is continuous and concave on $R^m_+$, and $\psi$ is upper semicontinuous and convex-valued on $R^m_+$.*

*Proof.* For $k$ in $R^m_+$, define $g(k)=\{\{k(t)\}_{t=0}^{\infty}:\{k(t)\}_{t=0}^{\infty}$ is a program from $k\}$. Also define $S'=\{\{k(t)\}_{t=0}^{\infty}:k(0)$ is in $R^m_+$, $(k(t),\ k(t+1))$ is in $\Omega$, for $t\geqq 0\}$. Then, clearly, $g$ is a correspondence from $R^m_+$ to $S'\subset S$. By the convexity of $\Omega$, the graph of $g$ is convex. To see that $g(k)$ is upper hemicontinuous, note firstly that $g(k)\neq\phi$, for all $k$ in $R^m_+$. Next, let $\{k^n(0)\}_{n=0}^{\infty}$ be a sequence of elements of $R^m_+$ converging to $\bar{k}(0)$ in $R^m_+$. Then, beyond some integer $\bar{n}$, $\|k^n(0)\|\leqq B(\bar{k}(0))+1$. Let $k^n\equiv\{k^n(t)\}_{t=0}^{\infty}$ be in $g(k^n(0))$ and so for $n\geqq\bar{n}$,

$\|k^n(t)\| \leqq B(\bar{k}(0))+1$, for all $t$ in $N$. Hence, by the Cantor diagonal process one can obtain a subsequence, $\{k^{n'}\}$, of $\{k^n\}$, and $\bar{k}=\{\bar{k}(t)\}_{t=0}^{\infty}$ in $S'$, such that for each $t$ in $N$, $\|k^{n'}(t)-\bar{k}(t))\| \to 0$, as $n\to\infty$, and so $d(k^{n'},\bar{k})\to 0$ as $n'\to\infty$. Since $\Omega$ is closed, $\{\bar{k}(t)\}_{t=0}^{\infty}$ is in $g(\bar{k}(0))$. This shows that $g(k)$ is compact for all $k$ in $R_+^m$, and $g$ is upper hemicontinuous on $R_+^m$. From the free-disposal assumption A.3, $g$ is monotone increasing.

For $k=\{k(t)\}_0^{\infty}$ in $S'$, define $f(k)=\sum_{t=0}^{\infty}\delta^t u(k(t),k(t+1))$. Then by the concavity of $u$, $f$ is concave on $S'$. Also, $f$ is upper semicontinuous on $S'$. To see this, let $\{k^n\}$ be a sequence in $S'$, $n=1,\ldots,$ such that $d(k^n,\bar{k})\to 0$ as $n\to\infty$. Then, for each $t\geqq 0$, $\|k^n(t)-\bar{k}(t))\|\to 0$ as $n\to\infty$. By the upper semicontinuity of $u$, we have

$$\limsup_{n\to\infty} u(k^n(t), k^n(t+1)) \leqq u(\bar{k}(t), \bar{k}(t+1)),$$

for each $t\geqq 0$. Thus

$$\begin{aligned}\limsup_{n\to\infty} f(k^n) &= \limsup_{n\to\infty} \textstyle\sum_{t=0}^{\infty} \delta^t u(k^n(t), k^n(t+1))\\ &\leqq \textstyle\sum_{t=0}^{\infty} \delta^t \limsup_{n\to\infty} u(k^n(t),\ k^n(t+1))\\ &\leqq \textstyle\sum_{t=0}^{\infty} \delta^t u(\bar{k}(t), \bar{k}(t+1)) = f(\bar{k}),\end{aligned}$$

noting that each expression in this string of inequalities is well-defined [by arguments similar to those used to prove the existence of an optimal program].

Now applying Theorem 2, $V$ is continuous and concave on $R_+^m$, and $\psi$ is upper hemicontinuous and convex-valued on $R_+^m$. □

The following result is well known as the 'principle of optimality' in dynamic programming, and is, therefore, stated without proof.

*Lemma 1.* (*a*) *If* $\{k(t)\}_{t=0}^{\infty}$ *is a program from* $\bar{k}$, *then* $V(k(t))\geqq u(k(t), k(t+1))+\delta V(k(t+1))$ *for* $t\geqq 0$.

(*b*) *If* $\{k(t)\}_{t=0}^{\infty}$ *is an optimal program from* $\bar{k}$, *then* $V(k(t))=u(k(t), k(t+1))+\delta V(k(t+1))$ *for* $t\geqq 0$.

(*c*) *If* $\{k(t)\}_{t=0}^{\infty}$ *is a program from* $\bar{k}$, *and for* $t\geqq 0$, $V(k(t))=u(k(t), k(t+1))+\delta V(k(t+1))$, *then* $\{k(t)\}_{t=0}^{\infty}$ *is an optimal program from* $\bar{k}$.

(*d*) *For* $k$ *in* $R_+^m$, $V(k)=\max_{(k,k')\text{ in }\Omega}\{u(k,k')+\delta V(k')\}$.

We now define the *optimal policy correspondence* $\mu$, by $\mu(k)=\{k'$ in $R_+^m:(k,k')\in\Omega,\ u(k,k')+\delta V(k')=V(k)\}$ for $k$ in $R_+^m$. Then $\mu$ is a correspondence from $R_+^m$ to $R_+^m$. For every $k$ in $R_+^m$, the set of elements $(k,k')$ in $\Omega$ is non-empty (by A.1, A.3), bounded [since $\|k'\|\leqq B(k)$, as noted above], and closed (by A.2). Furthermore, $u$ is upper semicontinuous on $\Omega$ (by A.5), and

$V$ is continuous on $R_+^m$ (by Proposition 1). So the correspondence, $\mu$, is non-empty on $R_+^m$.

*Proposition 2. The optimal policy correspondence, $\mu$, is upper hemicontinuous and convex-valued on $R_+^m$.*

*Proof.* If $\{k(t)\}_{t=0}^{\infty}$ is in $\psi(k)$, then $k(1)$ is in $\mu(k)$, by Lemma 1. Also, if $k(1)$ is in $\mu(k)$, then picking $k(t+1)$ as any element of $\mu(k(t))$ for $t \geqq 1$, we know by Lemma 1 that $\{k(t)\}_{t=0}^{\infty}$ is in $\psi(k)$. Thus, $\mu$ is upper hemicontinuous and convex-valued on $R_+^m$, since $\psi$ is upper hemicontinuous and convex-valued on $R_+^m$. □

*Remark 4.* (i) The optimal policy correspondence, $\mu(k)$, provides the set of states that it is optimal to go to, given that the initial state is $k$. The upper hemicontinuity of this correspondence is proved by Sutherland (1970) under the assumption that the utility function, $u$, is continuous on $\Omega$. Since $u$ is best interpreted as a reduced utility function (obtained by solving a maximization problem) it is possible to justify the assumption that $u$ is upper semicontinuous, but difficult to justify that $u$ is continuous on $\Omega$ [on this point see Peleg (1973), Khan and Mitra (1986), McKenzie (1986) and Dutta and Mitra (1986)].
(ii) Another application of our result is provided in Khan and Mitra (1986) in proving the existence of a non-trivial stationary optimal stock by applying the Kakutani fixed-point theorem. There, the correspondence, whose fixed-point turns out to be a non-trivial stationary optimal stock, can be shown to be upper hemicontinuous and convex-valued by applying a result like Theorem 2. For details, the reader is referred to Khan and Mitra (1986) or McKenzie (1986).

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